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MATHEMATICS FOR TEACHERS OF MATHEMATICS

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Abstract

The paper contains a sketch of a BSc Hons degree programme *Mathematics* (*for Mathematics Education*). It can be seen as a comment on Gardiner (2018) where he suggests that the current dire state of mathematics education in England cannot be improved without an improved structure for the preparation and training of mathematics teachers:

> *Effective preparation and training requires a limited number of national institutional units, linked as part of a national effort, and subject to central guidance.*
> *For recruitment and provision to be efficient and effective, each unit should deal with a significant number of students in each area of specialism* (*say* 20–100). *In most systems the initial period of preparation tends to be either*
>
> - *a "degree programme" of* 4–5 *years* (*e.g. for primary teachers*), *with substantial subject-specific elements, or*
> - *an initial specialist, subject-based degree* (*of* 3+ *years*), *followed by* (*usually* 2 *years*) *of pedagogical and didactical training, with some school experience.*

This paper suggests possible content, and didactic principles, of

> a new kind of "*initial specialist, subject-based degree*" designed for intending teachers.

This text is only a proof of concept; most details are omitted; those that are given demonstrate, I hope, that a new degree would provide a fresh and vibrant approach to education of future teachers of mathematics.

1. *Introduction*

This paper was originally prepared for the LMS Education Day 2017[†], with the theme **Teacher shortages in mathematics: how can HE mathematics departments help reverse the trend?**

The aim of the meeting was described as an attempt to formulate a cohesive approach to the problem:

> *University mathematics departments depend on teachers to prepare their own students, and they have an important role in training future generations of mathematics teachers. To do this effectively at a national level, it is critical that colleagues from*

2010 *Mathematics Subject Classification* 97B70.

[†]The LMS Education Day 2017, 25 May 2017, https://www.lms.ac.uk/events/education-day; Programme of the Day: https://bit.ly/2LPmrnn; a brief discussion: http://education.lms.ac.uk/2017/04/teacher-preparation-25-may-de-morgan-house-russell-square/.

> *across the sector understand the current state of Initial Teacher Training and the challenges that face teacher recruitment.* [...]

The issue of acute teacher shortages in school mathematics has featured in newspaper headlines for a number of years, and was brought under the spotlight again by a recent report *The teacher labour market in England: shortages, subject expertise and incentives*[†] from the Education Policy Institute[‡]:

> - *Teacher training applications are down by* 5%*, while training targets have been persistently missed in maths and science.*
> - *Exit rates have also increased, and are particularly high early on in teachers' careers. Only* 60% *of teachers remained in state-funded schools five years after starting. For 'high-priority' subjects like physics and maths, this 5-year retention drops to just* 50%. [...]
> - *Maths and most science subjects in particular struggle to attract highly-qualified teachers – with as little as half of teachers holding a relevant degree. Under* 50% *hold a relevant degree in maths and physics. These subjects, with the lowest proportion of highly-qualified teachers, are also those with the greatest recruitment and retention problems.* [...]
> - *In areas outside of London, just over a third* (37%) *of maths teachers* [...] *in poorest schools had a relevant degree. In more affluent schools outside of London, proportions are far higher for maths* (51%) *and chemistry* (68%). [...]
>
> *In London, differences in how highly-qualified teachers are represented are far smaller:*
>
> - *In maths, the proportion of teachers with a* [relevant] *degree ranges between* 40–50% *for all schools, regardless of deprivation level* [...]

The report of the Education Policy Institute misses some key issues which should be apparent to every practitioner or even observer of mathematics education; they are succinctly formulated by Gardiner (2018). Comparing England with other developing countries, he concludes that

> ***We know of no other system that pretends to train mathematics teachers by placing small groups of trainees at the mercy of teachers with no relevant ITE experience beyond being themselves teachers****, with much of the input being "generic" rather than subject-specific. England appears to be alone among developed nations in embracing such an approach.* (op. cit. p.5)

On the contrary,

> *Most systems prefer an initial period of training (in an HEI, Teachers' College, or Institute of Education),* followed by *a period which addresses some of the practical aspects of* teaching in school. *Other systems blend these two aspects of teacher preparation slightly differently –* ***but with overall responsibility being routinely vested firmly in the HEI****, or equivalent institution. In particular, each provider recruits, and uses their accumulated expertise to guide the progress of, a significant number of* potential teachers in their area of specialism.

Gardiner further suggests (op. cit.pp. 9–10) that the current dire state of mathematics education in England cannot be improved without efficient preparation and training of mathematics teachers, and that such a programme

> *... requires a limited number of national institutional units, linked as part of a national effort, and subject to central guidance.*

[†] https://epi.org.uk/publications-and-research/the-teacher-labour-market-in-england/; a pdf file: https://epi.org.uk/wp-content/uploads/2018/08/EPI-Teacher-Labour-Market_2018.pdf.

[‡] https://epi.org.uk/.

For recruitment and provision to be efficient and effective, each unit should deal with a significant number of students in each area of specialism (say 20–100). In most systems the initial period of preparation tends to be either

- *a "degree programme" of 4–5 years (e.g. for primary teachers), with substantial subject-specific elements, or*
- *an initial specialist, subject-based degree (of 3+ years), followed by (usually 2 years) of pedagogical and didactical training, with some school experience.*

This paper suggests possible content, and didactic principles, of

"an initial specialist, subject-based degree".

2. *Some principles behind the proposed Degree Programme*

This paper briefly outlines a possible Degree Programme, here tentatively entitled

BSc in Mathematics (for Mathematics Education)

The key idea of the proposed Degree Programme is to teach

- mathematics that is directly relevant to future secondary school teachers of mathematics,
- in the form and scope directly relevant to the teaching of mathematics,
- with a focus on the basics and the fundamentals,

but, crucially,

- at a level of demand which ensures confident problem solving skills.

Mathematical problem solving skills of graduates from the Programme are expected to go beyond procedural manipulation and include making proofs.

I suggest to use the term *Essential Mathematics* for description of an approach to teaching mathematics borrowed from language teaching. The *Essential English* method developed in 1930s by Michael West, Harold Palmer and Lawrence Faucett, was aimed at achieving full language fluency within a limited, but functional vocabulary.

In mathematics, *fluency* means

- ability to solve at least simpler problems,
- make proofs,
- see connections between various problems and known results,
- ability to reformulate a problem using different conceptual frameworks.

For future teachers of mathematics, *fluency* also means the ability to perform *didactic transformation* of teaching material (see Section 3.3).

In comparison with standard university mathematics courses, the proposed programme is limited in scope and stripped of superficial material that 95% of students do not understand anyway, but places an emphasis on problem solving skills. Currently, examination problems which require producing proofs of statements previously unknown to candidates are never offered in almost all British universities (Cambridge being perhaps a unique exception). The proposed degree programme brings proofs back as an integral part of the assessment.

Deeper understanding of basics and fundamentals together with problem solving skills should make graduates of the new Degree Programme attractive to employers more widely than the education sector – these are traits that are currently largely absent in most mathematics graduates.

In summary: streamlining and trimming the curriculum would allow us to put mathematical thinking back into university level mathematics education.

3. *What is needed and what is not needed*

3.1. *What can be ditched?* In comparison with mainstream mathematics courses, what can be ditched from a degree programme for further teachers of mathematics? This list is easy to compile:

> Algebraic Geometry, Algebraic Topology, Applied Complex Analysis, Approximation Theory and Methods, Asymptotic Expansions and Perturbation Methods, Chaos and Dynamical Systems, Commutative Algebra, Elasticity, Fluid Mechanics, Foundations of Finance, Fractal Geometry, Generalised Linear Models, Integral Equations and the Calculus of Variations, Introduction to Financial Mathematics, Linear Analysis, Markov Processes, Martingales with Applications to Finance, Matrix Analysis, Multivariate Statistics, Numerical Analysis, Option Theory, Principles of Mathematical Modelling, Quantum Computing, Random Models, Regression Analysis, Riemannian Geometry, Statistical Inference, Statistical Modelling in Finance, Time Series Analysis, Viscous Fluid Flow . . .

These courses are offered at a various universities in the UK; the list can be easily continued – but they all are irrelevant for future teachers of mathematics. Many mathematician colleagues would perhaps agree that most undergraduate students do not really understand them anyway.

3.2. *Guiding principles.* But what should be guiding principles for selecting mathematics courses which future teachers need?

As was already mentioned in Section 2, these courses should provide future teachers with mathematical tools for *didactic transformation* of learning material. To discuss that, we need a digression into some education theory.

3.3. *Didactic transformation.* A remarkably compact formulation of what makes mathematics education so special can be found in a paper by the prominent mathematician Hyman Bass:

> *Upon his retirement in 1990 as president of the International Commission on Mathematical Instruction, Jean-Pierre Kahane described the connection between mathematics and mathematics education in the following terms:*
>
> > In no other living science is the part of presentation, of the transformation of disciplinary knowledge to knowledge as it is to be taught (*transformation didactique*) so important at a research level.
> >
> > In no other discipline, however, is the distance between the taught and the new so large.
> >
> > In no other science has teaching and learning such social importance.
> >
> > In no other science is there such an old tradition of scientists' commitment to educational questions. (Bass (2005))

The concept of didactic transformation is fairly old and can be traced back to Auguste Comte (Comte (1852), preface):

> *A discourse, then, which is in the full sense didactic, ought to differ essentially from one simply logical, in which the thinker freely follows his own course, paying no attention to the natural conditions of all communication ...*
>
> *On the other hand, this transformation for the purposes of teaching is only practicable where the doctrines are sufficiently worked out for us to be able to distinctly compare the different methods of expanding them as a whole and to easily foresee the objections which they will naturally elicit.*

However, a Google Scholar search shows that the concept is used in the literature on mathematics teaching less widely than one would anticipate. The reason for that is that, in mathematics, didactic transformation is indeed a form of mathematical practice. Moreover, it is in a sense applied research since it is aimed at a specific application of mathematics: namely, mathematics teaching. It remains mostly unpublished, underrated and ignored because it is frequently confined to the early stages of course development or to the ephemera of classroom practice.

Didactic transformation could, and should be informed by advice from researchers in education and cognitive psychologists; however, methodologically it remains a part of hardcore mathematics. Indeed, returning to Compte's words:

> '... *the doctrines are sufficiently worked out for us to be able to distinctly compare the different methods of expanding them ...* (op. cit.)

We see that, in the context of mathematics teaching, the expressions 'to work out' and "expand' refer to purely mathematical activities: essentially, they mean 'to prove mathematically'. The situation with the words 'distinctly compare' is even more interesting: here we see in action the reflexive power of mathematics as a precise and flexible tool for study of the structure and function of mathematics itself.

3.4. *Didactic envelope of school level mathematics.* I suggest to use the term *didactic envelope* to describe a body of mathematics sufficient for understanding and **analysing** concepts, problems, and methods of a particular stage of mathematics education, and for preforming didactic transformation, if necessary, of the learning material.

In case of school mathematics (up to and including A Level), some grasp of the following disciplines is really useful.

- elementary set theory and logic
- Euclidean geometry in dimensions 2 and 3
- linear algebra in dimensions 2, 3, and 4
- real analysis
- **elementary complex analysis**
- linear differential equations
- mechanics

It does not mean that teachers should teach all that at school – but a good background will remove a slavish dependence on any particular curriculum and on learning materials provided by specific examination boards, which is so obvious in school practice in this country.

Please notice the emphasis on elementary complex analysis; this is really the core. I may expand on that point elsewhere.

3.5. *Target problems.* A good way to describe "intended learning outcomes' in a human-readable non-bureaucratic way is to give examples of problems that a graduate should be able to solve. Vladimir Arnold's famous *Mathematical Trivium*, Arnold (1991), is perhaps too demanding. Only one problem in Arnold's list is relevant to future school teachers of mathematics:

> Sketch the graph of the derivative and the graph of the integral of a function given by a free-hand graph.

In this text, I will sometimes use target problems as an indication of the intended content and level of proposed courses.

3.6. *Iconic problems.* These are almost an extinct species of school mathematics: elementary problems so unique and important that they do not survive tinkering and conversion into step-by-step scaffolded step-by-step examination problems. It is their uniqueness that makes a permanent irritant to those who only coach for success in standardised written examination.

A typical example is focal properties of a parabola. At A Level, considerable time is devoted to the calculation of gradients and tangent lines to basic curves such as the parabola. But what is the point in that activity if the focal properties, so beautiful and important for practical applications, are never mentioned?

In every course for teachers, iconic problems should be presented, solved, and analysed in the most prominent way.

A number of iconic problems can be found in our forthcoming book Borovik and Gardiner (2019).

3.7. *Universal methods.* In this country's mathematics education, it is forgotten that many problems of elementary mathematics can be solved by applying universal methods. Here are some examples; the list is not exhaustive.

- Every trigonometric identity can be proven either by applying Euler's formula, or Weierstass' substitution.
- There is an elementary universal method which, given a polynomial $f(x)$, produces a closed formula for the sum
$$f(1) + f(2) + f(3) + \cdots + f(n);$$
all these problems of the kind "prove that the sum of the first n triangle numbers equals ..." are, in effect, trivial.
- Most geometric constructions in Euclidean geometry are greatly simplified by using a combination of geometric loci and geometric transformations.
- Most statements in Euclidean geometry can be classified as affine or metric;
  * affine statements can be easily proven by vector algebra; a typical example: "three medians of a triangle intersect in one point";
  * metric statements may require more sophisticated metric considerations, but ultimately can be done by a brute force calculation.

A mathematics teacher would benefit from the ability to use these universal methods; this skill boosts confidence.

3.8. *Synergy and interconnectivity.* What makes the development of a cohesive study programme for *Mathematics* (*for Mathematics Education*) a challenging problem is the need to ensure as complete interconnectivity of learning material as possible. It is self-defeating to restrict the formal content of the programm, if the capacity thereby released is not used to make explicit every possible connection which exists between the facts and concepts of different mathematical disciplines.

Here I give a few examples of interconnectivity of themes and concepts in the syllabus listed in the next section.

- *Algebra of Polynomials and Rational Functions* + *Number Theory*: Lagrange's Interpolation Formula and the Chinese Reminder Theorem are one the same thing.
- *Algebra of Polynomials and Rational Functions* + *Number Theory*: an appropriately formulated version of Fermat's Last Theorem is true, and easy to prove, in the ring of polynomials over $\mathbb{C}$; the proof emphasises the role of uniqueness of factorisation, the point that is now believed to have been missed by Fermat[†].
- *Complex Analysis* + *Number Theory* + *Euclidean Stereometry* + *Analysis: Integration* + Non-Euclidean Geometry: stereographic projections of a circle and of a sphere produce Pythagorean triples, Weierstrass' substitution in integration of rational functions of trigonometric functions, the Riemann sphere, and much more, including some stuff that can be used in extension projects, for example, a transcription between the Beltrami-Klein and the Poincaré disk models of the hyperbolic plane.

3.9. *Emphasis on proofs and counterexamples.* Restricting the scope of mathematical content allows to free precious teaching time and to bring back proofs and counterexamples into undergraduate mathematics.

Ability to identify an error in a statement, argument, calculation, is a valuable skill for a future teacaher.

In all exam papers, students should be expected to produce proofs of simple statements previously unknown to them – or construct a counterexample.

4. *Tentative Syllabus: Year* 1

I wish to emphasise that this syllabus is a tentative proof of concept and requires further work for converting it into something practically usable. I would be happy to contribute to further development, but, at this stage, offer only a brief sketch.

[†]In Summer 2018, I had a chance to give a week long crash course on Numbers and Polynomials to schoolchildren at the Nesin Mathematics Village, http://www.nesinkoyleri.org/eng/, where elementary number theory and elementary theory of polynomials were developed in parallel and culminated in proving the two remarkable facts mentioned above. It worked.

SEMESTER 1

- Number Systems
- Algebra of Polynomials and Rational Functions
- Euclidean Planimetry
- Linear Algebra: Vectors and matrices in 2D and 3D. Lines and planes in 3D.
- Calculus of Elementary Functions
- Mathematical Workshop: Problem Solving Skills

SEMESTER 2

- Elementary Set Theory
- Number Theory
- Euclidean Stereometry
- Linear Algebra: Scalar product, orthogonality, quadrics in 2D and 3D.
- Calculus: Limits and derivatives.
- Mathematical Workshop: Problem Solving Skills.

At this level, all courses should be compulsory (to establish basic prerequisites for any subsequent options). All courses are 10 credits.[†]

To give some flavour, I comment on a few courses.

Number Systems describes the construction of canonical number systems: natural numbers $\mathbb{N}$, integers $\mathbb{Z}$, rational numbers $\mathbb{Q}$, real numbers $\mathbb{R}$ (via infinite decimal expansions), complex numbers $\mathbb{C}$, and, as an icing on the cake, a few words about the quaternions $\mathbb{H}$.

Algebra of Polynomials and Rational Functions – a few target problems give an indication of its content:

- Find the remainder when we divide $x^{2019}+1$ by x^2-1.
- Factorise x^4+1 as a product of two quadratic polynomials with real coefficients.
- Find a monic polynomial with integer coefficients and of minimal possible degree which has a root $\sqrt{2}+\sqrt{3}$. What are other roots of this polynomial?
- Given a polynomial $f(x)$, find a closed formula for the sum
$$f(1)+f(2)+f(3)+\cdots+f(n)$$
and prove it. This is much easier than most people think: see discussion in Section 3.7.

Calculus of Elementary Functions should give students fluency with functions, their graphs, identities linking rational, trigonometric, exponential, logarithmic functions. A few target problems:

- Sketch (don't "plot") the graph of
$$y=\frac{1}{x+1}+\frac{1}{x}+\frac{1}{x-1}.$$
- Give an example of a rational function with the graph of the shape shown on Figure 1.

[†]Only when I started writing this paper, I realised that over my 40+ years of teaching I have actually taught all these courses.

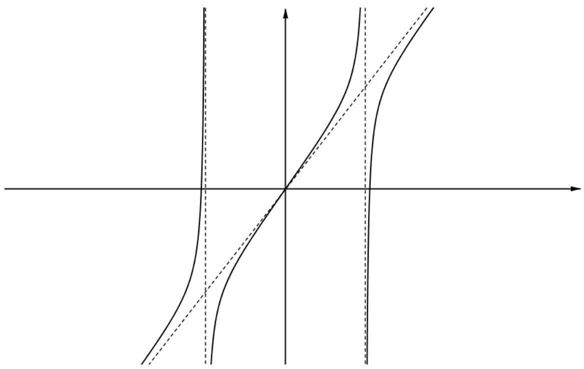

FIGURE 1.

- Take a random trigonometric identity from textbooks, say,

$$\sin 2\theta = 2\sin\theta\cos\theta$$

  and prove it.

- Sketch the graphs of the functions:
  (a) $y = \sin^{2019} x$
  (b) $y = \sqrt[2019]{\sin x}$

Linear Algebra Among many other things, one would like Linear Algebra in dimension 2 to include the classification of geometric transformations of the plane, and of plane conics; and in dimensions 3 and 4 to include an explanation of the intimate relations between quaternions and 3D vector analysis – relating the basic quaternion multiplications

$$\mathbf{i}\cdot\mathbf{j}=\mathbf{k},\ \mathbf{j}\cdot\mathbf{k}=\mathbf{i},\ \mathbf{k}\cdot\mathbf{i}=\mathbf{j},\ \mathbf{i}\cdot\mathbf{i}=\mathbf{j}\cdot\mathbf{j}=\mathbf{k}\cdot\mathbf{k}=-\mathbf{1},$$

to rotations of 3-dimensional space, linking pure quaternions $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ to 3D vectors and quaternion multiplication to the cross product in $\mathbb{R}^3$:

$$\vec{\mathbf{i}}\times\vec{\mathbf{j}}=\vec{\mathbf{k}},\ \vec{\mathbf{j}}\times\vec{\mathbf{k}}=\vec{\mathbf{i}},\ \vec{\mathbf{k}}\times\vec{\mathbf{i}}=\vec{\mathbf{j}},\ \vec{\mathbf{i}}\times\vec{\mathbf{i}}=\vec{\mathbf{j}}\times\vec{\mathbf{j}}=\vec{\mathbf{k}}\times\vec{\mathbf{k}}=\vec{\mathbf{0}}$$

This is fundamental to understanding spatial geometry and mechanics – and is best explained within a Linear Algebra framework.

Number Theory starts from basic assumptions about integers and division with remainder and is developed in the spirit of Niven et al (2008) as a sequence of simple statements with short proofs (a few lines), culminating in Euler's Theorem and the Chinese Remainder Theorem, with modular arithmetic developed along the way, with an explanation that the Chinese Remainder Theorem for integers and the Lagrange Interpolation Theorem are special instances of a more general result, to be proven in full generality somewhere in Year 2 or 3. Natural applications to Cryptography (RSA, the Miller-Rabin primality test, etc.) can be included.

Euclidean Geometry (both Planimetry and Stereometry) is to be taught in a simplified axiomatic approach, also with proofs (one possibility is to use a cute little book Pogorelov (1987)). In the exam, students will be expected to produce proofs of simple statements previously unknown to them. The course will benefit from systematic use of geometric visualisation software, CINDERELLA and GEOGEBRA.†

5. *Year 2.*

5.1. *Year 2: Compulsory Courses.*

SEMESTER 1

- Calculus: Integration, Multivariate Calculus
- Real Analysis
- Combinatorics
- Methodology of Mathematics Education I

SEMESTER 2

- Symmetry and Groups
- Mechanics
- Logic
- Methodology of Mathematics Education II

Each course is 10 credits (2 lectures a week).

A few comments on content of some of the courses.

Combinatorics should definitely include generating functions, linking it with power series and analysis.

- Another possible point of serious didactic value: Catalan numbers provide an example of *cryptomorphism*, expression of the same underlying mathematical structure in different mathematical languages and context. Catalan numbers enumerate dozens of quite natural combinatorial structures which, at a first glance, have nothing in common.

Mechanics A proper course of elementary mechanics, with an emphasis on frames of reference, conservation laws, symmetry, and scaling arguments. It should have a clear link with vector algebra, including cross product of vectors and use derivatives and basic linear differential equations. In real life applications of mechanics, the emphasis is on qualitative rather than quantitative problems; it is worth remembering that qualitative arguments in mechanics are in effect *proofs* – this gives a nice example of applications of proofs to the so-called real life. A few *iconic problems*:

- Give a rigorous mathematical explanation to the following rule frequently used in sailing:

  > *If a boat appears to be stationary with respect to some distant reference point or has the same compass bearing from your boat over a period of time then it is on a collision course with you.*

  You may assume that the boats are sailing with constant velocities, that is, with constant speeds and constant directions.
- When a moving snooker ball hits a stationary equal ball without loss of energy (elastic collision) and off-center, they start moving in perpendicular directions. Why?†

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†Comparative analysis of CINDERELLA and GEOGEBRA can be found in my paper Borovik (2012).

†This problem is an example of some of the most wondrous – but also some of the simplest and immediately open

- In the absence of external forces, the center of mass of a solid body (or of a system of material points) always moves along a straight line. Why?

5.2. *Optional courses available to Years* 2 *and* 3.

Mathematical courses: Discrete Mathematics, Mathematical Programming with Python, Game Theory, Non-Euclidean Geometries, Physics for Mathematicians, Projective Geometry.

Game Theory should perhaps cover about a quarter of the book DeVos and Kent (2016).

Optional courses on human aspects of mathematics (also available in Year 3) could also be very useful: Philosophical Logic, History of Mathematics, History of Mathematics Education, Philosophy of Mathematics, Communication of Mathematics; a project on History of Mathematics. These should not be seen as “soft options” – please see Appendix for a discussion of projects in History of Mathematics.

Optional Mathematics Education Courses (also available in Year 3): Mathematical Software and its Use in Learning and Teaching Mathematics: GeoGebra, Sage, Mathematica, etc.

6. *Year* 3

6.1. *Year* 3*: Compulsory Courses.*

Semester 1

- Probability
- Complex Analysis
- Methodology of Mathematics Education III
- School experience

Semester 2

- Statistics
- Non-Euclidean Geometries
- Methodology of Mathematics Education IV
- School experience

Notice that Probability and and Statistics are placed in Year 3 – for very good reasons.

Comments on some of courses:

Complex Analysis needs to contain just enough material to

- give the idea of the Riemann sphere and stereographic projection;
- construct models of hyperbolic geometry in the later course of Non-Euclidean Geometry;
- give some idea of analytic functions and conformal maps;

to observation – interactions between the fundamental principles of Physics and Mathematics: in this case, these are the Law of Conservation of Energy and the converse of Pythagoras Theorem.

- explain why the radius of convergence of the real power series expansion of
$$\frac{1}{1+x^2}$$
at $x = 0$ is bounded by poles at $z = +i$ and $-i$ in the complex domain;
- prove Euler's formula: $e^{it} = \cos t + i \sin t$ (there are several elegant proofs, see, for example, user02138 (2010) – and all of them are quite illuminating);
- discuss the use of stereographic projection and the Mercator projection[†] in cartography;
- prove that e and π are transcendental numbers.

6.2. *Year 3: Optional Courses.* Optional courses available in Year 3 only:

Algebra: Groups, Rings, Fields; Algebraic Cryptography, Coding Theory, Combinatorics and Graph Theory, Convex Geometry and Linear Programming, Differential Geometry, Entropy and Information, Metric Spaces, Partial Differential Equations, Peano Aritmetic, Probability and Information, Projective Geometry.

These courses go beyond the didactic envelope of school mathematics, but cover some neighbouring areas, including some applications of mathematics which can be usefully discussed in extension activities at school.

Comments on some of optional courses:

Partial Differential Equations are limited to the one-dimensional heat equation (with real-life applications such as basic models for permafrost) and the one-dimensional wave equation
$$\frac{\partial^2 \phi}{\partial t^2} - u^2 \frac{\partial^2 \phi}{\partial x^2} = 0$$
with further use of Fourier series solutions to explain musical scales and the difference in timbre of a guitar and a piano.

Peano Arithmetic develops arithmetic as an axiomatic theory, with proofs in the style of Edmund Landau's classical treatment Landau (2001): rigorous, but expressed in human language – and brings it up to division with remainder (thus acting as a prequel to earlier Number Theory). Huge number of accessible problems on proofs. Formally speaking, this is a zero prerequisite course.

Projective Geometry is an example of a course which could be enriched by obvious links to perspective as it is used in visual arts.

To develop a syllabus of this kind, and to fill it with appropriate learning materials is challenging, but can be achieved by a small consortium of mathematically strong universities. And there is every reason to believe that this can be done in a competent, efficient and cost effective way.

[†]The Mercator projection is a conformal map from the Riemann sphere onto a cylinder and is therefore nothing else but the uniformisation of the complex logarithm – a fundamental fact absent from most undergraduate courses of complex analysis.

7. *Conclusion*

Mathematics is not a sum of facts; it is a complex network of connections and analogies between facts, and, moreover, connections between connections and analogies between analogies. The *Essential Mathematics* approach to mathematical education of future teachers of mathematics is rooted in this principle.

Limiting the scope of the courses to the immediate didactic envelope of school level mathematics allows one to develop a plethora of connections between concepts and facts of elementary mathematics, to return proofs to the centre of mathematics education, and develop in future mathematics teachers complete mathematical fluency within elementary mathematics.

This text is only a proof-of-concept: full details are intentionally omitted. The draft will need to be modified, improved, and developed in much greater detail. However it is important first to gauge first the response of the mathematics community. But I hope the proposed degree programme provides a fresh and vibrant approach to undergraduate mathematics educations.

Appendix: Course projects on History of Mathematics

There is a view (not entirely unjustified) that projects in the history of mathematics are often a soft option for mathematics undergraduates. On the other hand, it is also widely acknowledged that project work can be a valuable part of a student's education.

Many mathematicians, not necessarily being specialists in the history of the subject, have no experience of how to design rigorous and challenging projects in the history of mathematics, and lack the time to do the required background preparation.

The proposed BSc degree *Mathematics for Mathematics Education* would benefit from a bank of carefully prepared – with no loss of rigour – projects in the History of Mathematics, suitable for second or third year undergraduates. Besides historic interest, the criteria for selection of themes will include:

- Presence of distinctive, challenging and clearly defined mathematical content.
- Accessibility of mathematical content to final year students.
- Scalability (which allows one to offer projects to students with varying degrees of preparedness for the work).
- Diversity and wide range of mathematical sub-disciplines covered (to help project supervisors to select, if they wish so, a project in a familiar their own area of mathematics).
- Availability and accessibility of original sources (preferably in a digitised form on the Internet); it is expected that most projects will involve work with original sources.

Here is a sample theme illustrating these principles: *History of Mathematical Induction.*

- Pierre de Fermat and his “infinite descent”;
- Augustus De Morgan and his introduction of mathematical induction;
- Dedekind’s and Peano’s axiomatisation of arithmetic;
- Edmund Landau’s book *Foundation of Analysis* Landau (2001);
- Georg Kantor and transfinite induction.

Works of Dedekind, Peano, and Kantor exist in English translations; De Morgan’s paper was written for *Penny Cyclopedia* (1838), a compendium of popular science for a mass readership; a digitised copy is available on the Internet.

Each project should have detailed guidance notes (for student and teacher) on how to approach the particular question, on which sources to use, and on how to write and present the final work, including advice on issues of copyright, intellectual property rights, and plagiarism. It will be normally expected that project work should be typeset in LaTeX (which provides unrivaled facilities not only for rendering mathematical formulae, but also for use of alphabets form other languages and for import of high quality graphics); appropriate LaTeX and LaTeX Beamer templates should be included as part of the package.

Acknowledgments

The author thanks Tony Gardiner for many useful discussions, and Tony Gardiner and Jane White for organising the LMS Education Day 2017. He also corrected a number of orthographic and stylistic errors in the text of this paper.

Draft specifications for course projects on History of Mathematics (Appendix I) were developed in a joint work with June Barrow-Green and Stephen Huggett who kindly allowed their use in this paper.

The author tested some of the didactic ideas presented in this paper in his lectures at the Nesin Mathematics Village in Şirince, Izmir Province, Turkey. My thanks go to Ali Nesin and to all volunteers and staff who have made the Village a mathematics research paradise, an oasis of proper mathematics education, and a garden of philosophy and arts Alladi and Nesin (2015), Karaali (2014).

Disclaimer

The views expressed do not necessarily represent the position of my employer or any other person, organisation, or institution.

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About the Author

I am an research mathematician but I have 40+ years of teaching experience at secondary school and university level in four different countries with four different education systems; since 1998 I am a Professor of Pure Mathematics at the University of Manchester. I teach a Foundation Studies (that is, Year 0) course in intermediate mathematics to 300+ students who were not successful in their A level mathematics, or did not take it at all, and for that reason I am an experienced end-user of of GCSE Mathematics.

I also have an interest in cognitive aspects of mathematical practice; see my book *Mathematics under the Microscope*, Borovik (2010), which explains a mathematician's outlook at psycho-physiological and cognitive issues in mathematics and mathematics education, and touches on many issues raised in this paper. Some of my papers on mathematics education can be found in my personal online journal/blog *Selected Passages From Correspondence With Friends*[†].

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