

NOTES ABOUT TEACHING MATHEMATICS AS
RELATIONSHIPS BETWEEN STRUCTURES:
A SHORT JOURNEY FROM
EARLY CHILDHOOD TO HIGHER MATHEMATICS

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*To my children, Elizabeth
and Theodore, who help me
understand mathematics better*

1. *Introduction*

No matter at which level, from mathematics related activities in kindergarten, to studying mathematical courses at university, many learners consider contents they are dealing with in a prescriptive way as a set of direct instructions. Following instructions they are able to complete some tasks. In most cases, however, such a performance is shadowed by their inability to deal with any other task that does not follow exactly the same prescriptive way. For example, if we have conditions A and B given and statement C is under the question mark, then under any change of conditions—what may happen with C , say, if A is replaced with a property A' ? Such a change may often cause insurmountable difficulties. As an outcome of this general tendency mathematics is seen by many learners as routine and useless, but still a complicated area of knowledge. Such an educational paradox, when the same kind of misunderstanding takes place in connection with completely different level of learning and does not depend on the age of the learners, deserves a separate discussion. The scope of this paper is bound within reflections about one teaching approach that may help to address this issue.

This approach focuses on understanding relationships between different structures. Under structures here I mean, in a wider context, any

mathematical object or set of objects and/or their properties, including any statement or visual representation, i.e. anything which can be considered from a mathematical point of view and related to a certain level of cognitive development. Seeing mathematics as relationships between structures is common for mathematicians. But, not for teaching mathematics yet. The idea to exploit this approach in teaching is not entirely new [3]. Its main aim is to make the “overall picture” clearer for learners. In other words, understanding what may happen beyond particular problems (situations) and how they may be connected with each other can bring further insight into details.

In this paper I will show how this teaching approach works through a number of examples, with many of them connected in different ways. Some of these connections may be regarded as new and unusual to some extent, while others are well-known due to their straightforward links with each other. All connections between different examples play an important role to justify this teaching approach and confirm its validity. All examples are discussed in sequence as the acquaintance with mathematics develops from early childhood through primary and secondary school to higher mathematics at undergraduate level.

I begin with two examples from early childhood to give evidence that the first encounter with mathematical content, not in the proper form, of course, may happen very early and can be highly productive. Then, I continue with examples from the primary schooling period. Again, to give evidence how fundamental mathematical concepts can be successfully introduced to a child’s mind. Next I will turn to non-trivial problems and theorems from secondary school and show how they can be influenced by what has been learnt in primary school and kindergarten. Finally, I discuss the formal side of higher mathematics with its connection and mutual impact to and from elementary mathematics.

2. *First steps into the special world: Mathematics is everywhere*

Teaching and research reports on early childhood and primary school mathematical activities are traditionally grouped around the number sense and numerical literacy. I deliberately leave these important conceptions untouched here and focus on the ways children produce their reasoning. This is amazing to see how logically, at one hand, and intuitively impeccable, at the other hand, a child’s mind can be towards

profound mathematical concepts without the actual mathematics being involved.

2.1. *Early childhood: Playing mathematics*

One of the approaches to develop mathematical skills and understanding in this age category is to avoid any mathematics! It may sound strange, but it does work! We cannot speak about the majority of mathematical concepts at this age. At least with most children. Exceptions, of course, exist. For example, one of the recent mathematical prodigies Terence Tao [2], or (looking back as far as around 100 years ago) Lev Landau who stated that he could not recall himself being unable to differentiate and integrate [5]. The examples I discuss here show that playing with non-mathematical “stuff” can be very promising from the mathematical point of view. In particular, it gives directions in which mathematical thinking of a child is ready to be extended and further developed.



FIGURE 1.

The first example is of visual form.

EXAMPLE 1 (A Picture from the Colouring Book). *A 3-year-old child was given a colouring picture. He was asked to colour whatever*

he wants in the picture. There was not any intervention into what he did. The output of the child's work is given in Figure 1.

Consider this colouring picture in detail. It shows clear evidence that the child's mind is, unconsciously, approaching a number of mathematical concepts. The main feature of the child's work is a well-balanced use of the same colours. It can be characterized in terms of such fundamental mathematical concepts as symmetry and one-to-one correspondence (including other types of correspondence between two sets). There are other features as well that can be developed into mathematical form. Bounded and non-bounded, as colouring has been performed within the body of an animation hero. This, also, can be related to potential readiness to perceive the concepts of infinity and convexity. The use of bordering lines where distinct colours are separated from each other can be characterized as pre-notional for open and closed sets. The list of features can be continued. . . My aim here, however, is to demonstrate that even at such a young age the child's brain is open to be connected with a diversity of mathematical ideas in the near future. Many of them are being already pre-incorporated into the child's cognitive schemes.

Next I turn to the second example to give further evidence on this matter.

EXAMPLE 2 (Tea Bags). There are two different brands of black tea served for lunch. But, tea bags are of different shapes. One brand is packed in small parallelepiped-like tea bags, while another in pyramid-like tea bags. A 3-year-old child knows that parallelepiped-like bags contain black tea. When he is told that black tea is in any pyramid-like bag too, his face expresses doubts about that. Black tea is associated with a parallelepiped shape in his view which is again the simplest one-to-one correspondence in child's mind. After getting reassurances that black tea is in pyramid-like bags as well the child makes a comment saying that black tea itself seems to be more important than how its packed shape looks.

The child's conclusion is far from trivial. Despite its vague meaning, for example what it really means here to be "more important", it does show the child's sensibility to operate with different statements at the pre-conceptual stage, the ability to arrange them according to some rules and, overall, readiness to produce some reasoning that corresponds to a certain level of cognitive development.

Before moving to the next section I would like to mention that a number of mathematical ideas and techniques originate from similar basic ideas as in Example 2. The difference is in the level of mathematics involved. For example, Peano's form of the remainder in Taylor's formula can be associated with Example 2—despite thousands of hours of learning mathematics separating them.

2.2. *Primary school: Reasoning makes sense*

As children grow up they become more experienced in learning mathematics. At this age we can already consider not only connections to some mathematical concepts as we did before, but also examples that introduce the basics of proof methods and problem-solving techniques with more applications to be considered onto higher level mathematics, secondary and tertiary. Two colours, red and green, from Example 3 are part of a proving process, while the UK pound bridges together two other currencies, the Australian dollar and the Euro in Example 4.

EXAMPLE 3 (Green and Red Cars). *I am driving a car with an 8-year-old child inside. I deliberately start naming all green cars we see around as being “red”. Red cars continue to be red ones. The child identifies the problem immediately and has a growing concern about what is happening with me. Soon we approach a street intersection with traffic lights ahead. The red light is on. I slow down and eventually stop waiting for the green light. When the green is on, I start driving and hear a huge sigh of relief. The conclusion follows: “You are joking!”.*

The child's reasoning is based on proof by contradiction: “If you are in trouble with colours, you won't drive across the intersection when the green light is on”.

From the child's point of view her reasoning does not have any relation to mathematics. From the teaching point it shows that many conceptual constructions in mathematics can be successfully introduced rather sooner than later. For example, the use of proof by contradiction presented here, as well as many other useful methods and structures, can be seen everywhere through elementary and higher mathematics. The teacher's task is to keep focus on them all the time—while moving from topic to topic, extending the content knowledge and improving problem-solving skills.

The next example is based on a simple idea which goes far beyond

primary mathematics context discussed here. This is a question from the Australian Mathematics Competition (AMC), Middle Primary Division paper.

EXAMPLE 4 (Dollar, Euro and Pound (AMC, 2010)). *Alex and his family plan to travel from Australia to England and then to France. They will need to change their money for each country. 100 Australian dollars converts to 40 English pounds, for England. 100 English pounds converts to 80 euros, for France. How many Australian dollars would be needed to get 120 euros?*

The idea to find the exchange rate between the Australian dollar and the Euro through their exchange rates with the English pound is straightforward in this example. More important, however, is a general principle that can be applied here—to establish possible relationships between mathematical structures A and B we may need to find an auxiliary structure C such that A and B are connected by means of C .

In the above example an auxiliary structure C —the English pound—is given explicitly, while for many other situations in learning mathematics at higher level such clarity can be a bit of luxury not necessarily available. More details about that will be given in the next section.

3. *Further steps into the special world or moving in-depth: Mathematics is still everywhere*

Secondary school mathematics and higher mathematics are the areas where learners deal with the proper mathematical ideas. Many of them are from the advanced level. However, most can be traced back to their origins on the elementary level. Examples given in this section highlight these connections and relationships between different structures. Skills to see such links and connections help to understand mathematics as one unique, though diverse area of knowledge.

3.1. *Secondary school: Enjoying non-trivial constructions*

Learning mathematics at this level is the time when understanding relationships between structures begins to bring more benefits. To some extent this outcome is due to the visible evidence of accumulation of mathematical knowledge since the time children had their first en-

counter with mathematics. Another contributing component is more regular attempts of learners to produce conscious efforts in seeing different contexts as part of one broader mathematical story, rather than unconnected parts of different stories.

The first example from this section demonstrates the simple idea from Example 4 being interpreted in terms of Euclidean geometry. The second example develops another interpretation of the same idea, now for inequalities. Both examples contribute to the diversity of links and interpretations between mathematical statements. In particular, how relationships between structures can be seen from more advanced, but still elementary level.

EXAMPLE 5 (Parallelogram Proof (Canada, 2011)). *Let $ABCD$ be a cyclic quadrilateral whose opposite sides are not parallel. Let X be the intersection of AB and CD , and Y be the intersection of AD and BC . Let the angle bisector of $\angle AXD$ intersect AD and BC at E and F respectively. Let the angle bisector of $\angle AYB$ intersect AB and CD at G and H respectively. Prove that $EGFH$ is a parallelogram.*

To work out this problem we need to prove that opposite sides of $EGFH$ are parallel. To do that we can try to identify some auxiliary lines (segments) which would be parallel to both opposite sides of the quadrilateral respectively. In terms of this problem the way that works is to show GF and EH are both parallel to AC , while EG and HF are both parallel to BD (Figure 2).

In this situation we deal with a non-trivial geometric problem where, roughly speaking, the role of English pound from Example 4 is taken by diagonals AC and BD . The difference is, however, that relations between two pairs of opposite sides of the quadrilateral $EGFH$ and two diagonals of the given quadrilateral $ABCD$ respectively have to be identified yet. The more unidentified links we have the more difficult task we may deal with. Example 8 from the next section provides more evidence about that.

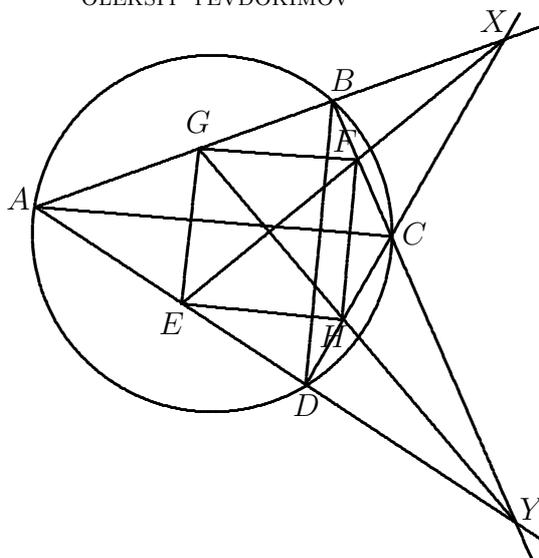


Figure 2.

Next I discuss two inequalities and analyze how they connect with each other and beyond that.

EXAMPLE 6 (Two Inequalities). *What may be common in attempting to solve the following inequalities?*

1. (China, 2004) Let a, b, c, d be positive real numbers such that

$$abcd = 1.$$

Prove that

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{1}{(1+d)^2} \geq 1.$$

and

2. (46th IMO, 2005) Let x, y, z be positive real numbers such that $xyz \geq 1$. Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{y^5 + z^2 + x^2} + \frac{z^5 - z^2}{z^5 + x^2 + y^2} \geq 0.$$

A general answer to the question above would not be difficult — solution methods for both inequalities. Indeed, keeping in mind the same principle as in Examples 4 and 5, but applied to inequality $A \geq B$, it could be worth to find a suitable estimation C that connects A and B , i.e. $A \geq C \geq B$. The details, though, are far from obvious. However, being aware about the former may help to focus on the latter. For the first inequality, applying twice the following inequality

$$\frac{1}{(1+x)^2} + \frac{1}{(1+y)^2} \geq \frac{1}{1+xy} \quad (3.1)$$

(it has to be identified yet!) leads to the required result. For the second inequality, applying the following inequality

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} \geq \frac{x^5 - x^2}{x^3(x^2 + y^2 + z^2)} \quad (3.2)$$

and its two cyclic equivalents (all three have to be identified yet!) leads to the required result too. The difference between the structure C in Example 5 and its counterparts from Example 6 is a higher level of difficulty to find C in Example 6 since the right-hand sides of (3.1) and (3.2) have never been part of given conditions or known information, while diagonals AC and BD in Example 5 may be seen as part of given conditions, at least implicitly in visual form. Of course, other solution methods can be used—it is not my aim to discuss them here.

3.2. Tertiary level: Working hard on a formal side

In this section I will show how relationships between structures can be seen in higher mathematics as well as through all levels. The knowledge learners acquire at this level include strict and formal definitions, rigour of proofs, axiomatic nature of many theories, etc. Some notions like infinity or convergence (in a broader sense) require special awareness to deal with. At one hand, the two examples from this section illustrate strong connections with different structures that have been learnt before. Some of them, like types of correspondence between sets—long before getting familiar with higher mathematics. At the other hand, a formal side of many conceptions is added into mathematical reasoning at this level which makes many ideas more diverse, more profound and, in many occasions, more elegant.

We have observed in Example 3 that the first acquaintance with proof by contradiction may happen very early. Next example shows how the same method works at much higher level.

EXAMPLE 7 (Erdős' Proof of the Infinity of Primes). *The infinity of primes is shown through the divergence of $\sum \frac{1}{p}$ where p takes all prime values.*

Assume $\sum \frac{1}{p}$ converges. Then, for any number, in particular for $\frac{1}{2}$, there must be a natural number k such that

$$\sum_{i \geq k+1} \frac{1}{p_i} < \frac{1}{2}.$$

By that number k the set of all primes is split into two sets

$$\{p_1, p_2, \dots, p_k\} \text{ and } \{p_{k+1}, p_{k+2}, \dots\}$$

called respectively *small primes* and *big primes*.

The special construction is created then to come to a contradiction. Two numbers N_b and N_s are introduced which are the numbers of positive integers $n \leq N$ that are divisible by at least one big prime, and have only small prime divisors respectively. Since $N_b + N_s = N$ for any N by definition, the proof that $N_b + N_s < N$ for a suitable N will lead to a contradiction [1].

The main feature of this proof is that the construction leading to a contradiction is not the result of direct conclusions from the assumption and given conditions. But, a separate original construction where assumption and some conclusions from it are taken into account on the way to a contradiction. Such situations are much harder to sort out. However, practice and experience being received with simpler constructions—looking as far back as to the first encounters with proof by contradiction as in Example 3—may help to deal with more complicated problems.

The fact that prime numbers form an infinite countable set also goes as far back as to Example 1 and similar situations when the first encounter with the concept of one-to-one correspondence, even between finite sets, takes place.

Finally, I pay attention to the famous Wallis' formula which is the first representation of π as a limit of the sequence with rational terms. This example, while it is itself interesting to see how different structures can be connected together to bring this amazing result, also has links to Examples 4, 5 and 6 to compare how similar ideas work on different levels.

EXAMPLE 8 (Example 8. Wallis' Formula).

$$\frac{\pi}{2} = \lim_{n \rightarrow \infty} \left(\frac{2n!!}{(2n-1)!!} \right)^2 \frac{1}{2n+1}.$$

Since

$$\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \begin{cases} \frac{(n-1)!!}{n!!} \cdot \frac{\pi}{2}, & \text{for even } n \\ \frac{(n-1)!!}{n!!}, & \text{for odd } n, \end{cases} \quad (3.3)$$

where $(n-1)!!$ means the product of all positive integers of the same parity not exceeding $n-1$, we can link together two sequences through their relationships to $\frac{\pi}{2}$

$$\left(\frac{2n!!}{(2n-1)!!} \right)^2 \frac{1}{2n+1} < \frac{\pi}{2} < \left(\frac{2n!!}{(2n-1)!!} \right)^2 \frac{1}{2n} \quad (3.4)$$

due to

$$\sin^{2n+1} x < \sin^{2n} x < \sin^{2n-1} x \quad \text{for } 0 < x < \frac{\pi}{2}. \quad (3.5)$$

Since the difference between two expressions in (3.4) converges to 0 as $n \rightarrow \infty$ due to

$$\frac{1}{2n(2n+1)} \left(\frac{2n!!}{(2n-1)!!} \right)^2 < \frac{1}{2n} \cdot \frac{\pi}{2}, \quad (3.6)$$

the Wallis' formula follows [4]

$$\frac{\pi}{2} = \lim_{n \rightarrow \infty} \left(\frac{2n!!}{(2n-1)!!} \right)^2 \frac{1}{2n+1}.$$

The key role in proving the Wallis' formula belongs to (3.4) which shows not only relationships between two sequences and $\frac{\pi}{2}$ in terms of inequalities, but also emphasizes the common nature of both sequences in terms of the limit concept. The latter is shown explicitly in (3.6). However, the starting point takes place in (3.5) which is not part of the given conditions, if the Wallis' formula is given as an exercise in either of the following forms:

$$(a) \quad \text{Find with proof } \lim_{n \rightarrow \infty} \left(\frac{2n!!}{(2n-1)!!} \right)^2 \frac{1}{2n+1}$$

or

$$(b) \quad \text{Prove } \lim_{n \rightarrow \infty} \left(\frac{2n!!}{(2n-1)!!} \right)^2 \frac{1}{2n+1} = \frac{\pi}{2}.$$

Despite being the answer, $\frac{\pi}{2}$ can be also seen here as an auxiliary structure C as in Examples 4, 5 and 6. It helps to connect two sequences together and if not given explicitly as part of the given conditions as in (a) above, $\frac{\pi}{2}$ and another sequence from (3.4) have to be identified yet which may bring more difficulties for learners due to more unidentified components and links.

4. Reflections

Many years ago one high school student (one of the best in his class—O.Y) complained to the author that everything related to quadratic equations is very boring due to its predictability, standard formulae for finding roots or use of Vieta formulae, etc. Later I have learnt, to my great surprise, that such a view was, if not common, then at least well-supported by many students from different schools. Of course, we can blame curriculum with focus on “wrong” objectives, lack of learners’ interest to studying mathematics, etc. But, such a misleading stereotype can be easily dismissed. For example, by the following problem from mathematical folklore:

A teacher wrote the polynomial $x^2 + 10x + 20$ on the board. After that each student, one after another, either increased by 1 or decreased by 1 either the coefficient of x or the constant term, but not both. Finally $x^2 + 20x + 10$ appeared on the board. One student is sure that a polynomial with both integer roots necessarily appeared in the process. Is she right or not?

This is a simple, though nice, problem based on reasoning, not on a straightforward calculation. Just one example from thousands of others that make a difference in understanding mathematics.

Later I came to believe that any misconception should not be destroyed by the teacher’s side in a forceable way. The teacher’s role is create correct conceptions and ideas enriching the textbook, such ones that have influence on learners’ thinking. Then, misconceptions will vanish on their own. The teaching approach described here makes a contribution in this direction. Indeed, understanding connections and links between different mathematical objects and statements is of great importance.

Furthermore, such understanding makes learning mathematics alive, dynamic and flexible with cognitive process taking a flagship position. The problem, however, is that this approach is not the easiest one. It is often hidden between the lines of many textbooks and requires some experience from teachers (to lead the process) and motivation from learners (to follow the process) to get the maximum of teaching and learning benefits. Unfortunately, the focus on relationships between structures is often neglected in everyday teaching and buried in routine work. For instance, if I look back at Example 4. I dare say that it is much more important to attract the attention of learners to these kinds of discussion than to follow a full solution. Without any emphasis on more general issues. Despite the fact that Example 4's simple and straightforward solution is based on general principle, such a general approach will hardly be visible by many learners of a certain age group without special notice from a teacher about that. Similar comments could be given to many other examples.

One may like or dislike the examples that have been brought to discussion here to highlight this approach. Some of them may look more persuasive than others. Different examples might suit even better for this purpose. I have attempted to show how unique and universal mathematics is from early childhood mathematical conceptions to advanced mathematical ideas of higher mathematics, and how this particular teaching approach might be used for developing learners' mathematical thinking. Summing up, this paper concerns learners' engagement to use reasoning at any level or at least provide them with a special mathematical environment to do so.

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